The Astrapi Technical Value Proposition

Dr. Jerrold Prothero

jprothero@astrapi-corp.com

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Technical Report

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Abstract

Astrapi® is the pioneer of spiral-based signal modulation, which opens an unexplored area for innovation at the core of telecommunications. Based on a generalization of Euler's formula, the foundational mathematics for telecom, Astrapi provides fundamentally new ways to design the symbol waveforms used to encode digital transmissions. Specific potential applications include improved performance in the presence of phase impairments, for instance due to multipath interference; greater resistance to interference from competing signals; and the ability to work well with power-efficient and cost-effective nonlinear power amplifiers. Most importantly, spiral-based signal modulation for the first time puts non-periodic signal modulation on a firm theoretical basis. Classical channel capacity theory implicitly assumes that signals are based on periodic functions. Non-periodic signal modulation opens a pathway to dramatically higher spectral efficiency, limited by hardware capabilities rather than solely by available bandwidth and signal-to-noise power ratio.

1. Introduction

Astrapi is the pioneer of spiral-based signal modulation. Simply put, this is the idea of basing telecommunication signal modulation on complex spirals, rather than the complex circles used by standard signal modulation techniques such as Quadrature Amplitude Modulation (QAM) and Phase-Shift Keying (PSK).

General background on spiral-based signal modulation, including an overview, engineering FAQ, mathematics, and applicable intellectual property, is available from the Astrapi website.[1]

Spiral-based signal modulation requires a fundamentally new approach to radio design. Why should anyone be interested in a long-term investment in this new telecommunications technology?

Because the telecommunications industry faces a severe long-term problem, which requires long-term solutions. The exploding demand for data transfer will be difficult to meet with existing or proposed solutions, potentially imposing costs in terms of user needs which are not met, profits which are not realized, and a significant drag on the world economy as a whole.

Astrapi believes that spiral-based signal modulation technology provides a fundamental solution to the fundamental problem of the Information Age, which is helping to ensure rapid, high-volume data transfer to meet exponentially growing demand.

Why do we believe this? At a high level, for three reasons.

1. *Spirals are a wide-open field for innovation.* There has been no prior research on spiral-based signal modulation. We know that spiral-based signal modulation is no worse than the current state-of-the-art, because traditional circle-based signal modulation is a subset of spiral-based signal modulation. The extra capabilities of spirals open up new possibilities to be exploited for efficient communication.
2. *There are specific applications where spiral-based modulation provides potential benefits.* These include situations limited by phase noise, coherent noise from interfering channels, and where efficient nonlinear power amplification is required (for reasons of power efficiency or equipment cost).

[1] http://www.astrapi-corp.com/technology/

3. *Spirals are the right way to transition from periodic to non-periodic signal modulation, which allows for greatly increased spectral efficiency.* The classical theory of channel capacity implicitly assumes that signals are based on periodic functions. Spirals, and the associated mathematics, opens a path to exploiting the potentially much higher spectral efficiency of non-periodic signal modulation.

Each of these points is discussed in more detail below.

2. An Open Field for Innovation

It is surprising but true that the idea of basing signal modulation on complex spirals, rather than the traditional complex circles, has not previously received research attention from industry or academics. Consequently, Astrapi holds very broad foundational patents, with no prior art, on the idea and techniques for spiral-based signal modulation.

Clearly, a circle is the special case of a spiral whose complex amplitude is constant. Conversely, spirals can do something circles cannot, which is to change their complex amplitude continuously over time.

It is common knowledge in any field that extra design parameters open up the possibility of much better products. Spiral-based signal modulation opens this door for telecommunications.

3. Specific Applications of Spiral-Based Signal Modulation

What are specific examples of communication applications in which spiral-based signal modulation may out-perform traditional circle-based signal modulation? The following three broad areas are intended to be illustrative, rather than comprehensive.

1. *Applications where phase noise is significant, for instance due to multipath interference or clock recovery problems.* Spirals have two potential advantages: lower dependence on phase information for a given data rate, since we can also store information in spiral amplitude variation; and the ability to use spiral amplitude variation for time alignment.
2. *Applications in which coherent noise from interfering channels is significant.* While it may not be initially apparent, the symbol

waveform design space becomes very much larger when symbol waveforms are constructed from complex spirals, rather than complex circles. This gives us the ability to make spiral-based signals look very different from other interfering signals, and therefore more easily detected in the presence of coherent noise.

3. *Applications benefiting from efficient nonlinear power amplification.* Many current signal modulation techniques require linear power amplification, whereas the physics of power amplification (and therefore of efficiency and cost) favors nonlinearity. We can use the wide complex amplitude variation information that spirals provide on a per-symbol basis to characterize and correct for nonlinear distortion.

The common theme in these applications is that the extra design space that spirals provide can be used to compensate for problems that frequently appear with traditional signal modulation.

4. The Spectral Efficiency Advantage of Spiral-Based Signal Modulation

The most powerful implication of spiral-based signal modulation is that it provides the most natural transition from periodic to non-periodic signal modulation, and that making this transition in principle allows spectral efficiency to be dramatically increased beyond what is usually thought to be possible.

"Periodic signals" are those that return regularly to the same sequence of amplitude values. This is equivalent to requiring that the frequency domain for the signal does not vary with time. Which in turn is equivalent to the statement that the frequency domain is constructed from frequency components (most simply but not necessarily sinusoidals) which have constant scaling coefficients.

Of course, all real signals are non-periodic, particularly at the boundary between symbols. However, as we shall see below, Shannon's foundational proof of the Shannon-Hartley channel capacity law made an implicit simplifying assumption that signals are periodic, an assumption carried through into later work in channel capacity theory.

The assumption of periodicity is a reasonable approximation for traditional signals whose symbol waveforms are each composed of sinusoidals with constant scaling coefficients. However, spirals are constructed from sinusoidals with continuously-varying coefficients, and are therefore non-periodic on an instant-by-instant basis. An assumption of periodicity is therefore a very poor match for spiral-based signal modulation.

Astrapi has proved that if the assumption of periodicity is removed, then non-periodic channel capacity can in principle be dramatically higher than the limit specified by the Shannon-Hartley law.[2] However, the proof requires new and unfamiliar non-periodic mathematical machinery (the generalized Euler's formula and spiral-based polynomial decomposition). This can pose a barrier for some readers.

There is also an existing literature on exceeding the classical Shannon limit in the presence of nonlinearity, notably with the recent introduction of regenerative mapping.[3] However, nonlinear channel capacity theory is unfamiliar to most practitioners in the telecommunications industry.[4]

The remainder of this paper is intended to bring the discussion onto ground more familiar to a broader technical audience, which is the classical channel capacity literature. We seek to establish a modest but important point: that, if we examine what has actually been proven, rather than what is widely believed to have been proven, there is nothing in classical channel capacity theory inconsistent with the idea that non-periodic channel capacity can in principle be dramatically higher than the limit specified by the Shannon-Hartley law.

4.1 What Shannon Proved (And Did Not Prove)

The foundational work for communication theory was provided by Claude Shannon in two papers: the first, "A mathematical theory of communication"[5], in which he

[2] J. Prothero (2012). The Shannon law for non-periodic channels. *Astrapi Technical Report R-2012-1.* Available from http://www.astrapi-corp.com/technology/white-papers/

[3] M. Sorokina, S. Turitsyn (2014). Regeneration limit of classical Shannon capacity. *Nature Communications* 5:3861 doi: 10.1038/ncomms4861. Available from http://www.nature.com/ncomms/2014/140520/ncomms4861/full/ncomms4861.html

[4] Nonlinearity and non-periodicity are of course distinct but related ideas. The spirals introduced by Astrapi are both. We believe that it is more useful to focus on the non-periodicity than the nonlinearity of spiral-based signal modulation, since it is natural to think of the spectral efficiency advantage that spiral-based signal modulation provides as arising from continuous non-periodic variations in the frequency domain.

[5] C. Shannon (1948). A mathematical theory of communication. *Bell System Technical Journal* **27**, 623-656.

introduced the formal framework for describing information transfer; and the second, “Communication in the presence of noise”[6], in which he proved what we now call “Shannon’s law”, or the “Shannon-Hartley law”, for the information capacity of a bandlimited noisy channel. For current purposes, the second of these is the critical paper, which we will examine below.

Reduced to its basics, Shannon established proofs for two key questions.

1. *What is the maximum rate at which independent amplitude values can be emitted from a transmitter, given an available bandwidth B?* Answer: the Nyquist rate of $f_N = 2B$. This fact is known as the “Sampling Theorem”.
2. *Given a sufficiently long sequence, how many bits can be encoded per amplitude value, assuming signal power S and noise power N?* Answer: $\log_2(\sqrt{1 + \mathrm{S/N}})$.

Multiplying these two results together, the maximum rate at which information can be transmitted is given by the “channel capacity” of

$$C = f_N log_2\left(\sqrt{1 + \mathrm{S/N}}\right) \quad (1)$$

Shannon chose to express the Nyquist rate as $2B$, and to remove the square root from the logarithm, giving his channel capacity formula in the more elegant (if less transparent) form

$$C = B log_2\left(1 + \frac{S}{N}\right) \quad (2)$$

It is important to understand, however, what mathematical tools Shannon used in his proof of the Sampling Theorem. Every set of tools makes certain things possible, while putting other things outside of the scope of the proof.

The essential idea behind Shannon’s proof of the Sampling Theorem is that sampling at the Nyquist rate in the time domain is sufficient to fully specify the Fourier series coefficients for the signal’s frequency domain, and therefore to fully determine the signal. Sampling above the Nyquist rate therefore provides no additional information.

[6] C. Shannon (1949). Communication in the presence of noise. *Proceedings of the IRE* **37**(1), 10-21.

However, in order for this proof to work, the frequency domain has to be stationary (non-time varying), which is equivalent to assuming that the signal is periodic.

If the assumption of periodicity is removed, Shannon's proof of the Sampling Theorem simply fails, as does the Shannon-Hartley law which depends upon it. This implies that Shannon's proof of the Shannon-Hartley law does not necessarily apply to signals based on non-periodic functions such as spirals.

The fact that using Fourier series coefficients, as Shannon did, assumes periodicity is not controversial: it is mentioned in every introduction to the subject.

Shannon was no doubt aware that his channel capacity law was limited to periodic signals, and probably thought this point too obvious to be worth mentioning. Certainly, in the context of what was possible or useful for 1940's communication engineering, there was little point to discussing non-periodic signals.

Unfortunately, since it was published Shannon's proof has been more admired than studied, and the fact that it is limited to periodic signals has been lost from the common understanding. This fact is not mentioned even in modern textbooks, at a time when non-periodic signals are certainly realizable.[7]

It is perfectly reasonable to ask whether Shannon's assumption of periodicity was very limiting. While non-periodic signals are not covered by the Shannon-Hartley law, does this leave much room in practice to increase channel capacity through non-periodicity? Or is non-periodicity simply a messy version of periodicity which has little new to offer?

A formal answer to this question is available, proving that non-periodicity in principle allows channel capacity to be dramatically increased beyond the periodic Shannon-Hartley limit.[8] However, in keeping with the theme of this paper, which is to stick with familiar mathematics, let us instead make two intuitive arguments.

The first intuitive argument that non-periodicity has something new to offer is as follows. If sampling at the Nyquist rate is required to fully reconstruct a signal with a stationary (periodic) frequency domain, perhaps a higher sampling rate will be necessary to reconstruct a signal with a frequency domain which can in principle

[7] See, for instance, J. Proakis & M. Salehi (2008). Digital communications. 5th Edition, McGraw-Hill 74, 365-367.
[8] *Ibid.*, 2.

vary at each instant in time. If so, the Shannon-Hartley law will be a poor approximation to non-periodic channel capacity.

A second and more detailed intuitive argument arises from considering why periodicity matters when using the Fourier series, as Shannon did in his proof of the Sampling Theorem.

"Fourier's trick" for moving from the time domain to the frequency domain is to assume that the time domain of a function $f(t)$ can be represented by sinusoidals whose frequencies are integer multiples of each other, then to use the fact that such sinusoidals are orthogonal to each other to isolate the coefficient for each sinusoidal.

Clearly, this works fine in the simple case where $f(t) = 1$. But what if $f(t)$ is a more general function? For the integral of $f(t)\cos(nt)\sin(nt)$ to equal zero, as it must for Fourier's trick to work, $f(t)$ must provide exactly the same weight for values of t where $\cos(nt)\sin(nt)$ is positive as it does for values of t where $\cos(nt)\sin(nt)$ is negative; and similarly for all other orthogonal pairs.

This constraint on the allowable functions $f(t)$ (which is equivalent to assuming the periodicity of $f(t)$) is obviously very restrictive. It raises a strong suspicion that more general non-periodic functions $f(t)$ might allow a wider scope for symbol waveform design, greater noise resistance, and higher spectral efficiency. In principle while using the same amount of bandwidth.

Viewed from this light, Shannon's Sampling Theorem could be seen as proving that periodic functions are so constrained and self-correlated that a periodic analog waveform is determined by a remarkably small number of amplitude values (provided by sampling at the Nyquist rate).

As shown above, Shannon's proof of the Sampling Theorem, on which the Shannon-Hartley law depends, does not necessarily apply to non-periodic signals. However, it is certainly possible in principle that later researchers might have resolved this limitation.

To the best of my knowledge, no researcher since Shannon has both recognized that Shannon's proof of the Sampling Theorem was limited to periodic signals, and claimed to have provided a more general proof of the Sampling Theorem showing that sampling at the Nyquist rate is sufficient to reconstruct non-periodic signals.

Such an extension would be necessary to prove that the Shannon-Hartley channel capacity law applies to non-periodic signals.

Nonetheless, more sophisticated mathematical tools have appeared since Shannon's foundational work, notably prolate-spheroidal functions and the Karhunen-Loeve transform. It is reasonable to inquire whether these techniques implicitly extended the scope of the Shannon-Hartley law to non-periodic signals. This topic is addressed below.

4.2 Later Research

In classical channel capacity theory after Shannon, the most familiar names are Landau, Pollak, Slepian and Wyner. This section briefly reviews their work from the perspective of whether they extended the Sampling Theorem, and thus the Shannon-Hartley law, to cover non-periodic signals.

It is important to observe that none of them made the claim of having done so.

Wyner, for instance, writes of Shannon's proof that "there is no question as to the meaning and validity of the capacity formula"[9], and makes no attempt to extend its scope to non-periodic signals.[10]

The question at hand is therefore whether the more sophisticated mathematical techniques they employed implicitly did extended the scope of the Shannon-Hartley law.

In a series of papers,[11] [12] [13] Landau, Pollak and Slepian introduced the prolate spheroidal wave functions as a means to precisely describe time limited and bandlimited functions, an area in which Shannon relied on the intuition of perfect

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[9] A. Wyner (1966). The capacity of the band-limited Gaussian channel. *Bell System Technical Journal* 659-395. Available from http://www.alcatel-lucent.com/

[10] See also A. Wyner (1976). A note on the capacity of the band-limited Gaussian channel. *Bell System Technical Journal* 343-346. Available from http://www.alcatel-lucent.com/

[11] D. Slepian & H. Pollak (1961). Prolate spheroidal wave functions, Fourier analysis and uncertainty – I. *The Bell System Technical Journal* 43-63. Available from http://www.alcatel-lucent.com/

[12] H. Landau & H. Pollak (1961). Prolate spheroidal wave functions, Fourier analysis and uncertainty – II. *The Bell System Technical Journal* 65-84. Available from http://www.alcatel-lucent.com/

[13] H. Landau & H. Pollak (1962). Prolate spheroidal wave functions, Fourier analysis and uncertainty – III. *The Bell System Technical Journal* 1295-1335. Available from http://www.alcatel-lucent.com/

bandlimiting.[14] The third paper in this sequence can be seen as a more sophisticated approach to the Sampling Theorem than Shannon presented.

However, as the titles of their papers would suggest, Landau, Pollak and Slepian's techniques rely on Fourier analysis, and specifically on the ability to describe the time domain with a stationary frequency domain, which carries forward Shannon's assumption of signal periodicity.

In the first of these papers, Slepian and Pollak[15] identified a set of functions (the prolate spheroidal wave functions) as the eigenfunctions of the Fourier transform over finite intervals. As eigenfunctions simplify many problems, this provides a very useful tool for describing functions that are limited in both time domain and frequency domain. It allows the idea of bandlimiting to be discussed with considerable rigor. However, the underlying assumption of the paper is that functions possess Fourier transforms (Section II) in the sense that the time domain can be described by a frequency domain that does not vary over time. This assumes periodicity, and therefore anything built upon this paper also assumes periodicity.

In the second paper in the sequence, Landau and Pollak[16] apply the theory of the first paper to the study of time limited and bandlimited signals. They show that the common understanding that a function cannot be simultaneously confined tightly in the time and frequency domain can be described precisely in terms of the amount of energy contained in a frequency band, and discuss applications.

In the third paper in the sequence, Landau and Pollak[17] provide a more sophisticated proof of Shannon's Sampling Theorem, making use of the prolate spheroidal wave functions. They show that a signal can be accurately approximated by a number of basis functions equal to twice the signal's bandwidth times the signal's duration, which implies that sampling at the Nyquist rate is sufficient to reconstruct the signal.

However, this proof inherits the periodicity assumption of the first paper in the sequence as described above, and therefore does not extend the Sampling Theorem to cover non-periodic signals.

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[14] Slepian provided interesting comments in his later John von Neumann Lecture. D. Slepian (1983). Some comments on Fourier analysis, uncertainty and modeling. *SIAM Review,* **25**(3) 379-393.
[15] *Ibid.,* 11
[16] *Ibid.,* 12
[17] *Ibid.,* 13

Wyner,[18] who as mentioned above makes no claim to have extended Shannon's proof of the Sampling Theorem to non-periodic signals, inherits periodicity assumptions from both Shannon's Sampling Theorem and Slepian and Pollak's prolate spheroidal wave functions. His application of the Karhunen-Loeve expansion also contains an implicit periodicity assumption.

The Karhunen-Loeve Transform (KLT)[19] is in a sense a generalization of the Fourier transform. Fourier analysis seeks to resolve a signal into sinusoidals; the KLT instead identifies a set of weighted orthogonal functions which ideally represent the signal + noise input sequence in terms of the signal + noise self-correlation matrix. However, the KLT assumes time-invariant coefficients for its orthogonal functions, which repeats in a more general context the Fourier assumption of periodicity.[20]

Nor, to briefly cover two other important authors, do either Van Trees[21] [22] or Harmuth[23] claim to extend Shannon's Sampling Theorem to cover non-periodic signals.

In short, to the best of our knowledge there is no researcher after Shannon who either deliberately or implicitly extended Shannon's proof of the Sampling Theorem to cover non-periodic signals. In the absence of such a proof, the Shannon-Hartley law cannot be considered to apply to non-periodic signal modulation.

The fact that periodic signal modulation has dominated both theoretical and applied attention in telecommunications, to the extent that the potential benefits of non-periodic signal modulation are generally unrecognized, probably largely reflects the available mathematical tools.

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[18] *Ibid.,* 9

[19] A very readable introduction to the KLT is provided in C. Maccone (2012). Mathematical SETI: statistics, signal processing, space missions. Springer. Maccone devoted 15 years to applying the KLT to special relativity.

[20] And the assumption of orthogonality is itself less general that the spirals-based signals introduced by Astrapi.

[21] H. Van Trees (2013). Detection estimation and modulation theory Part 1 – detection, estimation and filtering theory. 2nd Edition, John Wiley & Sons.

[22] H. Van Trees (1971). Detection estimation and modulation theory part 2 – nonlinear modulation theory. John Wiley & Sons.

[23] H. Harmuth (1972). Transmission of information by orthogonal functions. 2nd Edition, Springer-Verlag.

Certainly, the practical engineering difficulties associated with non-periodic signal modulation are sufficient to explain why it is not already the basis for everyday telecommunications. But that its very possibility is not familiar to otherwise well-informed engineers and researchers reflects a deeper issue.

The Whorfian hypothesis in linguistics states that one's language determines one's conception of the world. The underlying mathematical language for the telecommunications industry arises from Euler's formula, introduced in the mid 18th Century. Euler's formula generates the sinusoidals used in standard signal modulation, and Fourier analysis which is used to analyze these signals. This is the mathematical language for thinking about periodic functions.

It is true that the windowed Fourier transform, wavelets and sparse representations[24] open an approach to non-periodicity. But they do not present non-periodicity natively in its own language, as something with an inherent structure, elegance and power of its own.

Such a native mathematical language for non-periodicity arises from generalizing Euler's formula.[25] And, consistent with the Whorfian hypothesis, it is from this new non-periodic mathematical language that Astrapi's spiral-based signal modulation arose.

5. Conclusion

Astrapi's introduction of non-periodic spiral-based signal modulation opens a new frontier in telecommunications innovation. Fundamental problems require fundamental solutions. The telecommunications industry faces a fundamental problem in the exponential growth of data transmission demand. Astrapi's technology provides a fundamental solution through new tools for symbol waveform design, making possible dramatic improvements in spectral efficiency.

[24] S. Mallet (2009). A wavelet tour of signal processing: the sparse way. 3rd Edition, Elsevier.
[25] See http://www.astrapi-corp.com/technology/white-papers/